Problem Set IV: due Friday, March 16

1) Consider a fluid in hydrostatic equilibrium with a vertical entropy gradient $\partial S/\partial z < 0$. Take $\underline{g} = -g\hat{z}$.

a) Starting from the basic equations, derive the growth rate of ideal Rayleigh-Bernard instability. You will find it helpful to relate the density perturbation $\tilde{\rho}/\rho_0$ to the temperature perturbation $\tilde{T}/T_o$ by exploiting the fact that the instability develops slowly in comparison to the sound transit time across a cell. Relate your result to the Schwarzschild criterion discussed in class.

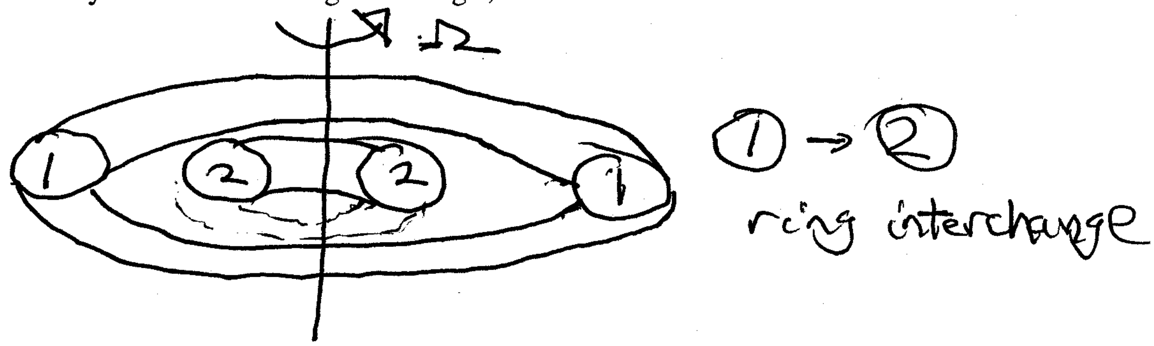
b) Now, include thermal diffusivity (χ) and viscosity (ν) in your analysis. Calculate the critical temperature gradient for instability, assuming $\chi \sim \nu$. Discuss how this compares to the ideal limit. What happens if $\nu > \chi$?

2) Now again, consider the system of Problem 2, now immersed in a uniform magnetic field $\underline{B} = B_0\hat{z}$.

a) Assuming ideal dynamics, use the Energy Principle to analyze the stability of a convection cell of vertical wavelength k_z. Of course, $k_z L_p >> 1$, where L_p is a mean pressure scale length. What is the effect of the magnetic field? Can you estimate how the growth rate changes?

b) Now, calculate the growth rate using the full MHD equations. You may assume $\underline{\nabla} \cdot \underline{V} = 0$. What structure convection cell is optimal for vertical transport of heat when B_0 is strong? Explain why. What happens when $B_0 \rightarrow \infty$? Congratulations - you have just derived a variant of the Taylor-Proudman theorem!

3) Consider a rotating fluid with mean $\underline{V} = r\Omega(\mathrm{r})\hat{\theta}$. Your task here is to analyze the stability of this system to interchanges of 'rings', i.e.



In all cases, assume $\underline{\nabla}\cdot\underline{V} = 0$ and $k_\theta = 0$, so the interchange motions carry no angular momentum themselves and the cells sit in the *r*-*z* plane.

a) At the level of a "back-of-an-envelope" calculation, calculate the change in energy resulting from the incompressible interchange of rings (1) and (2). Note that $E = L^2/2mr^2$ and that the angular momentum L of an interchanged ring is conserved, since $k_\theta = 0$. From this, what can you conclude about the profile of $\Omega(r)$ necessary for stability? Congratulations - you have just derived the Rayleigh criterion!

b) Now, calculate the interchange growth rate by a direct solution of the fluid equations. You may find it helpful to note that for rotating fluids in cylindrical geometry:

$$\frac{\partial V_r}{\partial t} + \underline{V}\cdot\underline{\nabla}V_r - \frac{V_\theta^2}{r} = \frac{-1}{\rho}\frac{\partial P}{\partial r},$$

$$\frac{\partial V_\theta}{\partial t} + \underline{V}\cdot\underline{\nabla}V_\theta + \frac{V_r V_\theta}{r} = \frac{-1}{\rho r}\frac{\partial P}{\partial \theta},$$

$$\frac{\partial V_z}{\partial t} + \underline{V}\cdot\underline{\nabla}V_z = \frac{-1}{\rho}\frac{\partial P}{\partial z}.$$

Show that you recover the result of part (a).

c) Compare and contrast this interchange instability to an incompressible Rayleigh-Taylor instability. Make a table showing the detailed correspondences.

4) *Drift-Alfven Waves*

a) Derive three coupled reduced fluid equations for $\phi, A_\parallel, n$. You may assume $T_e >> T_i$ and electrons are isothermal. Include a strong $\underline{B}_0 = B_0\hat{z}$ and $\langle n \rangle = \langle n(r) \rangle$.

b) Show that in the limit where $A_\parallel$ is negligible, you recover the Hasegawa-Wakatani system. Calculate the dispersion relation for drift instability in this system. Discuss your result in the limit $k_\parallel^2 v_{Th}^2/\omega v > 1$.

c) Calculate the quasi-linear particle flux related to part b), above.

d) Show that if $\hat{n}$ and $d\langle n \rangle/dr$ are negligible, you recover reduced MHD. What waves are present in this system? Discuss and derive the dispersion relation.

e) Derive the dispersion relation for the full 3 equation system. Discuss how drift and shear-Alfven waves couple for $k_\parallel^2 v_{Th}^2/\omega v > 1$.

5) Consider magnetic buoyancy interchange instabilities as discussed in class. Assume entropy stratification is neutral, so $dS_0/dz = 0$. Take η small, but non-zero.

a) Use quasilinear theory to calculate the vertical flux of magnetic intensity. Since, $\Gamma \sim -\partial_z \ell n(\langle B \rangle/\rho)$, show that Γ may be written as

$$\Gamma = -D\frac{\partial\langle B \rangle}{\partial z} + V\langle B \rangle.$$

Calculate D, V. Interpret your result. For $\rho = \rho_0(z)$. What profile corresponds to the zero flux state?

b) What is the origin of the pinch velocity V? Explain its significance.

c) As a related example, consider evolution of the particle density according to

$$\partial n/\partial t + \nabla \cdot (n\underline{V}) = 0.$$

Take $n_0 = n_0(x)$, $\underline{B} = B_0(x)\hat{z}$ and $\underline{V} = -\nabla\phi x\hat{z}/B_0(x)$.

Show that density evolution can be related to the incompressible advection of the field n/B:

$$\partial n/\partial t + \underline{V}_{eff} \cdot \underline{\nabla}(n/B) = 0$$

where $\underline{\nabla} \cdot \underline{V}_{eff} = 0$.

Show that the mean field equation for $\langle n \rangle$ obeys:

$$\frac{\partial}{\partial t}\langle n \rangle = \frac{\partial}{\partial x} D \frac{\partial}{\partial x}\left(\frac{\langle n \rangle}{\langle B \rangle}\right)$$

where we took $\langle n/B \rangle \cong \langle n \rangle / \langle B \rangle$. Discuss the zero flux state here. What are its implications for the density profile?

Re-write the mean field equation as

$$\frac{\partial \langle n \rangle}{\partial t} = -\frac{\partial}{\partial x}\left[-D\frac{\partial \langle n \rangle}{\partial x} + \mathrm{V}\langle n \rangle\right].$$

Relate D and V, here. Under what circumstances will V be inward, i.e. *up* the density gradient?

d) Relate the results of parts b.), c.) here. What is the lesson?

Congratulations! You have just developed the basics of TEP pinch theory!